INFLUENCE OF INITIAL MICROSTRESSES ON THE MACROSCOPIC STRAIN OF POLYCRYSTALS

PMM Vol. 32, №5, 1968, pp. 908-922
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(Received June 10, 1968)

A quasistatic theory of plasticity of polycrystalline bodies which took approximately into account the inhomogeneity in the plastic deformation and its accompanying elastic microdeformations, was proposed in [1]. This theory is extended below to the case when there are still initial elastic microstresses in the body which do not destroy the statistical isotropy and homogeneity of the body in its initial state, as well as the microstresses caused by the inhomogeneity in plastic deformation. Such an extension of the theory is of indubitable interest since initial microstresses inevitably originate in the formation of polycrystals by cooling of a liquid metal. An indication of the essential role of these initial stresses in the plastic deformation process is encountered in the literature ([2], say), however, insofar as the authors know no attempt has yet been made to describe this role theoretically (i. e. the influence of the initial microstresses on the connection between the macroscopic stresses and the plastic strains). As will be seen later, taking account of the initial elastic microstrains and their corresponding stresses induces noticeable quantitative and qualitative corrections in the theory of plasticity of polycrystals.

1. Some supplements to the paper [1]. A modification of plasticity theory based on the conception that the macroscopic plastic deformation can be expressed approximately as the arithmetic mean of a finite number of elementary plastic deformations to which different values of the local yield point correspond, was proposed in Section 1 of [1]. In this modification, all the random quantities, yield point, stresses and plastic deformations, were averaged over a volume enclosing a sufficiently large quantity of crystal grains.

The mentioned modification of the theory is extended in Section 2 of the same paper to the model of a polycrystal consisting of an infinite number of elements, in conformity with which the sums are replaced by integrals. Hence, the yield point, the stresses and the plastic deformations are considered as functions of some scalar random parameter x, in substance, the intensity of the dry friction force tensor T, whose realization will be denoted by τ, will be the fundamental random scalar parameter of the problem. The density $p(\tau)$ of the distribution T should be known and is considered identical at all points of the polycrystal because of the assumption of its statistical homogeneity. However, parameters x and ξ, uniquely connected with, and introduced instead of T and τ in [1], are selected so that integration is performed within the limits $0 < \xi < 1$. The connection between τ and ξ is defined by Expression

$$\xi = \int_0^\tau p(\tau)\, d\tau \qquad (0 < \xi < 1) \tag{1.1}$$

i. e. ξ is an integral function of the probability distribution of T.

If T and τ are inserted in place of x and ξ in the relationships (2.4) of [1], these latter become

$$\langle E_{ij}^p\rangle = \int_0^\infty \varepsilon_{ij}^p(\tau)\, p(\tau)\, d\tau, \qquad \langle \Sigma_{ij}\rangle = \int_0^\infty \sigma_{ij}(\tau)\, p(\tau)\, d\tau \tag{1.2}$$

$$\langle \Sigma_{ij}\rangle - \sigma_{ij} = m\,(\varepsilon_{ij}^p - \langle E_{ij}^p\rangle) \tag{1.3}$$

$$\tau_{ij} = \sigma_{ij} - \int_0^\infty c(\tau, \tau')\, \varepsilon_{ij}^p(\tau')\, p(\tau')\, d\tau' \tag{1.4}$$

$$de_{ij}^p = \tau_{ij}\, d\lambda, \qquad \sqrt{\tau_{ij}\tau_{ij}} = \tau \tag{1.5}$$

Here E_{ij}^p, (ε_{ij}^p) is the random plastic deformation tensor; Σ_{ij}, (σ_{ij}) the random stress tensor; T_{ij}, (τ_{ij}) the random tensor of dissipative forces of resistance to plastic deformations.

The notations for the realizations of these random tensors are indicated in parantheses.

Integration in these formulas is performed between the limits $0 \leqslant \tau \leqslant \infty$ in conformity with the fact that the random scalar quantity τ is always positive. Its probability density is a physical characteristic ot the polycrystal, being a measure of the spread in the local yield points.

Let us turn attention to the fact that the last two Eqs. (1.4) and (1.5) are written to the realizations. They express the local flow law of the polycrystal. The quantity τ will be constant in the integration of the differential relations (1.5). Moreover, let us note that integration with respect to ξ in [1] (exactly as integration with respect to τ in (1.2)–(1.4)) means averaging over the set of realizations rather than over the volume. Meanwhile, the modification of the theory presented in Section 1 of [1] is based on averaging over the volume. Hence, the extension of this theory to a model with an infinity of elements is based substantially on the ergodic hypothesis; this circumstance was not stipulated in [1].

2. Introduction to the theory of initial microstresses. Let us assume that there is a random field of initial elastic microstrains E_{ij}°, (ε_{ij}°) in the polycrystal, which does not destroy the macroscopic homogeneity and isotropy of the body in its initial state. Because of the requirement that the body be statistically homogeneous, this random tensor should be a stationary function of the coordinates, and because of the requirement of statistical isotropy, all its principal directions should be equally likely, from which there follows that the probability density of the six-dimensional random variable E_{ij}° can be a function only of the invariants ε_{ij}°. Only the deviator part of the tensor E_{ij}° is hence of interest for the sequel since precisely this part exerts influence on the development of the plastic deformations. Consequently, and also considering that the global and deviator parts of E_{ij}° are independent random variables, we shall henceforth assume that E_{ij}° will be the deviator (in order not to have to introduce a special notation for its deviator part). Under the stipulations made, the density of the distribution E_{ij}° can be a function only of the second and third invariants of ε_{ij}° and its mathematical expectation $\langle E_{ij}^\circ\rangle$ should be zero. However, it is natural to make the additional assumption that not only all the principal directions of E_{ij}° are equally likely, but all kinds of initial microstrains are also equally likely since there is no foundation to expect that there is a preference for any of them (for example, at least for pure shear or tension). Then the single possibility remains that the probability density of E_{ij}° is a function of its second invariant (its intensity)

$$F(\varepsilon_{ij}^{\circ}) = F(\varepsilon^{\circ}), \qquad \varepsilon^{\circ} = \sqrt{\varepsilon_{ij}^{\circ}\varepsilon_{ij}^{\circ}} \tag{2.1}$$

Let us note that the positive random variable ε° has as upper bound the inequality

$$\varepsilon^{\circ} \leqslant \tau / 2G \tag{2.2}$$

in which G is the elastic shear modulus (strictly speaking, the local elastic shear modulus. However in the interests of the considered approximate theory in which averaging with respect to crystallite orientations is assumed to have been performed in advance, G should be identified with the effective shear modulus of the polycrystal).

The meaning of the inequality (2.2) is that no initial microstresses with an intensity exceeding the local yield point can originate during cooling of the metal. Hence, depending on the local situation ε° can turn out to equal $\tau / 2G$, or even a lower value. However, to simplify the theory it is admissible to replace the inequality (2.2) by the equality

$$\varepsilon^{\circ} = \tau / 2G \tag{2.3}$$

As is evident, such an assumption is directed towards exaggerating the influence of the initial microstresses on the macroscopic picture of the plastic deformation, since the intensity of the initial elastic microstrains is here estimated by its upper limit. It is easy to note that there should be no elastic strain domain in such a theory: any arbitrarily small change in loading made in any direction should cause plastic deformation since according to (2.3) the dry friction force turns out to be equilibrated by the initial microstresses.

The concept of flow boundaries (yield surface) will thereby be lacking in such a theory in the sense in which it is ordinarily understood in courses or plasticity theory. However, as is known, the concept of a flow boundary (yield point) will be the result of idealization of the experimentally observed picture of plastic deformation.

In fact, plastic deformations are observed for any loading changes, and the more accurate the test formulation, the more rapidly are they detected. It is known that the engineering tensile yield point is defined as that stress for which the plastic elongation achieves an 0.2% magnitude.

The fact that the yield point must be determined, in practice, by assigning the magnitude of the plastic deformation emphasizes the conditionality of this concept. Therefore, the absence of the yield point concept in a theory based on the equality (2.3), will not be a flaw, but will more rapidly indicate the approach of theory to experiment.

It should be emphasized that the yield point and flow boundary concepts can be introduced even in this modification of the theory if the threshold of the intensity of plastic deformation is given, i.e. that intensity starting with which the plastic deformations are taken into account, and below which are neglected. It is natural that the dimensions and shape of the flow boundaries obtained here will depend on the magnitude of this provisional threshold, as has been observed experimentally [3].

In addition to the modification of the theory based on (2.3), some other simplified modifications of the theory will also be considered below. In particular, a modification based on the assumption that $\varepsilon_0 = \text{const}$, $\tau = \text{const}$, where $\varepsilon_0 < \tau / 2G$ will be investigated.

It turns out that there is the concept of a flow boundary in the usual sense in such a theory, where corner points are formed on this boundary during plastic deformation.

The acceptance of the assumption (2.3) established a single-valued interrelation between two random scalar parameters, the intensity of the dry friction force τ and the intensity of the initial elastic strains, where the density of the distribution E° should be identical to the density of the distribution $T/2G$. Formally, the introduction of the initial microstrain tensor into a theory expressed by the set of Eqs. (1.2)–(1.5) is not difficult, however, the problem hence turns out to be dependent not only on the random scalar T, but also on the random tensor E_{ij}°, which makes quite complicated the solution of the equations for specific kinds of loading.

It is hence expedient to utilize (1.2)–(1.5) henceforth in a somewhat simplified form, by replacing (1.4) by one of the two relationships

$$T_{ij} = \langle \Sigma_{ij} \rangle - \alpha E_{ij}^{p} \tag{2.4}$$

$$T_{ij} = 2G\,[\langle E_{ij} \rangle - \beta E_{ij}^{p}], \quad E_{ij} = E_{ij}^{p} + E_{ij}^{e} \tag{2.5}$$

These relationships have the essential advantage over (1.3) that the ε_{ij}^{p} is not under the integral sign.

As has already been remarked in [1], the application of (2.5) corresponds to the Besseling theory [4] extended to a model with an infinite number of elements. This modification of the theory describes the picture of the macroscopic plastic deformations correctly, although it does not permit taking account of some fine effects observed under cyclic loadings. As regards (2.4), in its possibilities it is approximately equivalent to (2.5).

The subsequent exposition will be oriented primarily towards utilization of the modification of the theory with (2.4) in place of (1.4). Taking account of the initial microstrains this equation is written as follows:

$$T_{ij} = \langle \Sigma_{ij} \rangle - \alpha E_{ij}^{p} - 2GE_{ij}^{\circ} \tag{2.6}$$

Here it has been taken into account that the initial elastic microstrains can contribute to, or conversely, hinder overcoming local dry friction, depending on the direction of the effective stresses. The scalar coefficient α in (2.6) is henceforth assumed constant. Equation (2.6) can be reduced to an identical form to (2.4)

$$T_{ij} = \langle \Sigma_{ij} \rangle - \alpha E_{ij}^{*p}, \qquad E_{ij}^{*p} = E_{ij}^{p} + \frac{2G}{\alpha} E_{ij}^{\circ} \tag{2.7}$$

It hence follows that taking account of the initial elastic microstrains in the above-mentioned formulation of the problem can be formally substituted by introducing the field of initial plastic deformations

$$E_{ij}^{\circ p} = \frac{2G}{\alpha} E_{ij}^{\circ} \tag{2.8}$$

The set of Formulas (2.7), (1.5), (2.3) forms a modification of the quasistatistical theory of plasticity of micro-inhomogeneous bodies taking account of the initial elastic microstrains. In order to be able to apply these formulas, in particular, in order to derive relationships between the macroscopic stresses $\langle \Sigma_{ij} \rangle$ and the macroscopic plastic deformations $\langle E_{ij}^{p} \rangle$ from them, it is however necessary to know the distribution law of the random tensor E_{ij}°.

3. Densities of the distributions E_{ij}° and T. As has been remarked, the random initial microstrain tensor should possess spherical symmetry because of the requirement of initial isotropy and homogeneity of the polycrystal, i.e. its probability density will be a function just of its intensity. In combination with the tensor linearity of the equations of the theory under consideration, this latter makes utilization of the vector interpretation of Il'iushin [5] henceforth possible and convenient since all the formulas presented below are invariant in five-dimensional vector space.

In place of the six components of the deviator E_{ij}°, which are interconnected by the linear dependence $E_{kk}^\rho = 0$, let us introduce the five independent quantities

$$E_1^\circ = \sqrt{3/2}\, E_{11}^\circ,\ E_2^\circ = \sqrt{2}\,(E_{22}^\circ + 1/2\, E_{11}^\circ)$$

$$E_3^\circ = \sqrt{2} E_{12}^\circ,\ E_4^\circ = \sqrt{2} E_{33}^\circ,\ E_5^\circ = \sqrt{2} E_{13}^\circ \tag{3.1}$$

considered as components of a five-dimensional vector whose length is

$$\sqrt{E_1^{\circ 2} + E_2^{\circ 2} + \cdots + E_5^{\circ 2}} = \sqrt{E_{ij}^\circ E_{ij}^\circ} = E_0 \tag{3.2}$$

We shall treat the quantity ε_k° as the realization of the random five-dimensional vector E_k°. Because of the properties of the random tensor E_{ij}° described earlier, the mathematical expectation of this five-dimensional vector is zero, and its probability density will be a function just of the length of the vector of its realization ε_0(3.2). Therefore [6], the vector E_{ij}° is normally distributed, i.e. its probability density is determined by

$$p(\varepsilon_k) = \frac{1}{(2\pi)^{5/2}} \frac{1}{a^5} \exp \frac{-\varepsilon_0^2}{2a^2} \tag{3.3}$$

in which a^2 is the variance, which is identical for all E_k°.

Having assigned the distribution law of the components of the five-dimensional vector E_k°, we can also calculate the distribution law of its length

$$E^\circ = \sqrt{E_1^{\circ 2} + E_2^{\circ 2} + \cdots + E_5^{\circ 2}} \tag{3.4}$$

To do this, let us utilize the known expression for the density of the distribution of a random variable χ^2 with k degrees of freedom ([7], p. 192). Putting $k = 5$ therein, and going from the case of unit variance to a^2 variance, we obtain

$$p(\varepsilon_0^2) = \frac{1}{2^{5/2}\Gamma(5/2)\, a^5} \exp \frac{-\varepsilon_0^2}{2a^2} \tag{3.5}$$

From this formula, which yields the density of the distribution ε_0^2, results the following expression for the density of the distribution ε_0.

$$p(\varepsilon_0) = \frac{2}{3} \frac{1}{\sqrt{2\pi}} \frac{\varepsilon_0^4}{a^5} \exp \frac{-\varepsilon_0^2}{2a^2} \tag{3.6}$$

which extends the known Rayleigh law governing the distribution density of the length of a random plane vector whose components are independent random normally distributed variables, to five-dimensional vectors. A detailed exposition of the theory of random multidimensional vectors with spherical symmetry can be found in [6].

For the theory based on the assumptions (2.3), the distribution density of the intensity of the dry friction force $p(\tau)$ is defined by the same formula (3.6)with a^2 replaced by

$$b^2 = (2G)^2 a^2 \tag{3.7}$$

i.e.

$$p(\tau) = \frac{2}{3} \frac{1}{\sqrt{2\pi}} \frac{\tau^4}{b^5} \exp \frac{-\tau^2}{2b^2} \tag{3.8}$$

The information presented above permits evaluation of the mathematical expectation and variance of any random variables which are functions of the random tensor E_{ij}° and the random scalar T. Hence, although (3.8) was obtained in close connection with assumption (2.3), it should be utilized even in the other modifications of the theory, thereby considering that the tensor of the dry friction forces has neither preferred orientations of the principal axes, nor any more probable than other values of the Lode parameter. Under the conditions of assumption (2.3), the invariant ε_0 will not be a random independent parameter (since it is connected with τ by means of (2.2)). This latter must be taken into account in calculating the mathematical expectations or variances of random variables which depend on T and E_{ij}^0. In the general case ε_0 should be considered an independent random variable in the domain bounded by (2.2).

It follows from (3.8) that the mathematical expectation of the intensity of the dry friction force (which can be interpreted as a macroscopic yield point of a polycrystal free from initial microstresses) is determined by

$$\langle T\rangle = \frac{2}{3}\frac{1}{\sqrt{2\pi}}\frac{1}{b^5}\int_0^\infty \tau^5 \exp\frac{-\tau^2}{2b^2}\,d\tau = \frac{16}{3\sqrt{2\pi}}b \approx 2.11b \qquad (3.9)$$

i.e. turns out to be proportional to the mean-square deviation of the random scalar T (the intensity of the dissipative forces resisting plastic deformation).

This circumstance is constantly utilized practically by metallurgists who raise the yield point of alloys by doping their crystal lattices with inclusions whose purpose is to increase the spread of the local yield points relative to their mean value, i.e. to increase the variance b^2.

4. Macroscopic one-dimensional loading of a body with initial microstrains. Let us apply the considered theory to the case of one-dimensional strain by utilizing here the following Eqs.:

$$d\varepsilon_{ij}^p = \tau_{ij}\,d\lambda, \qquad \tau_{ij} = \langle\Sigma_{ij}\rangle - \alpha\varepsilon_{ij}^p, \qquad \sqrt{\tau_{ij}\tau_{ij}} = \tau \qquad (4.1)$$

and considering that there are initial plastic strains $\varepsilon_{ij}^{p\circ}$. It is expedient to replace the system (4.1) by an equivalent system with five independent unknowns ε_k^p

$$d\varepsilon_k^p = \tau_k\,d\lambda, \quad \tau_k = \langle\Sigma_k\rangle - \alpha\varepsilon_k^p \qquad (4.2)$$

Following Il'iushin [5] here

$$\varepsilon_1^p = \sqrt{3/2}\,\varepsilon_{11}^p, \qquad \varepsilon_2^p = \sqrt{2}\,(\varepsilon_{22}^p + 1/2\,\varepsilon_{11}^p)$$
$$\varepsilon_3^p = \sqrt{2}\,\varepsilon_{12}^p, \qquad \varepsilon_4^p = \sqrt{2}\,\varepsilon_{23}^p, \qquad \varepsilon_5^p = \sqrt{2}\,\varepsilon_{13}^p \qquad (4.3)$$

The quantity $\langle\Sigma_k\rangle$ is expressed in terms of $\langle\Sigma_{ij}\rangle$ by means of formulas analogous to (4.3). It is easy to confirm (if the equality $\varepsilon_{jj}^p = 0$ $\langle\Sigma_{jj}\rangle = 0$ is taken into account here) that the system (4.1) will actually be a corollary of (4.2).

Furthermore, let us introduce another change of variables

$$\rho_k = \frac{\alpha\varepsilon_k^p}{\tau}, \qquad s_k = \frac{\langle\Sigma_k\rangle}{\tau}, \qquad t_k = \frac{\tau_k}{\tau} \qquad (4.4)$$

Then (4.2) will reduce to the following simpler form:

$$d\rho_k = t_k d\gamma, \quad t_k = s_k - \rho_k \qquad (t = \sqrt{t_1^2 + t_2^2 + \cdots + t_5^2}) \qquad (4.5)$$

where

$$\gamma = \frac{\alpha}{\tau}\lambda \qquad (4.6)$$

Let us apply (4.5) to the case of a macroscopically homogeneous loading. Hence

$$s_k = 0 \quad \text{for } k \neq l, \qquad s_l = s \qquad (4.7)$$

Substituting (4.7) into (4.5), we obtain

$$d\rho_l = (s - \rho_l)\,d\gamma, \quad d\rho_k = -\rho_k\,d\gamma \quad (k \neq l), \quad t = \sqrt{(s - \rho_l)^2 + \rho_*^2} \qquad (4.8)$$

In this latter formula

$$\rho_*^2 = \sqrt{\rho_1^2 + \rho_2^2 + \cdots \rho_5^2} - \rho_l^2 = \rho^2 - \rho_l^2 \qquad (4.9)$$

From the second formula of (4.8) there results that

$$\rho_k = \rho_{k0} l^{-\gamma} \tag{4.10}$$

where ρ_{k0} is the value of ρ_k at $\gamma = 0$, i.e. at the time of the beginning of uniaxial plastic loading. It follows from (4.10) that

$$\rho_* = e^{-\gamma} \sqrt{(\rho_{10}^2 + \rho_{20}^2 + \cdots + \rho_{60}^2) - \rho_l^2} = \rho_{*0} e^{-\gamma} \tag{4.11}$$

from which

$$d\rho_* = -\rho_* d\gamma \tag{4.12}$$

Now, integration of the system (4.8) reduces to integrating a system of two equations

$$d\rho_l = (s - \rho_l)\, d\gamma, \quad d\rho_* = -\rho_* d\gamma \tag{4.13}$$

under the additional condition

$$(s - \rho_l)^2 + \rho_*^2 = 1 \tag{4.14}$$

Putting $\rho_l = \rho_{l0}$, $\rho_* = \rho_{*0}$ therein, we obtain two values

$$s_{10} = \sqrt{1 - \rho_{*0}^2} + \rho_{l0}, \qquad s_{20} = -\sqrt{1 - \rho_{*0}^2} + \rho_{l0} \tag{4.15}$$

bounding the domain of values of s

$$s_{10} > s > s_{20} \tag{4.16}$$

within whose limits the strains are elastic, i.e. $\rho_k = \rho_{k0}$.

The system (4.13) is easily integrated. Its solution taking (4.15) into account is

$$\rho_e = \{s - \rho_{l0} - \operatorname{th}[s - s_0 + \operatorname{arth}(s_0 - \rho_{l0})]\} \times$$
$$\times \chi[(s - s_0) \operatorname{sign} s_0] + \rho_{l0} \tag{4.17}$$
$$\rho_* = \left\{\frac{1}{\operatorname{ch}[s - s_0 + \operatorname{ar\,th}(s_0 - \rho_{l0})]} - \rho_{*0}\right\} \times \chi[(s - s_0) \operatorname{sign} s_0] + \rho_{*0}$$

where $\chi(x)$ is the Heaviside function

$$\chi(x) = 1 (x > 0), \ \chi(0) = 0 \ (x < 0)$$

It follows from the second formula of (4.17), (4.10) and (4.11) that

$$\rho_k = \left\{\frac{\rho_{k0}}{\rho_{*0} \operatorname{ch}[s - s_0 + \operatorname{ar\,th}(s_0 - \rho_{l0})]} - \rho_{k0}\right\} \times \chi[(s - s_0) \operatorname{sign} s_0] + \rho_{k0} \ (k \neq l) \tag{4.18}$$

The solution (4.17), (4.18) combines both cases: $s_0 = s_{01}$ and $s_0 = s_{0,2}$ under the condition that none of the boundary values is zero. The last case is possible, and will be essential later. Let us recall that it was assumed earlier (in the previous section) that the intensity of the initial random microstresses at each point of the body equals the local yield point. Corresponding to this is the equality $\rho_0^2 = 1$ for which

$$s_{01} = |\rho_{l0}| + \rho_{l0}, \ s_{02} = -|\rho_{l0}| + \rho_{l0} \tag{4.19}$$

Hence $s_{02} = 0$ for $\rho_{l0} > 0$ and $s_{01} = 0$ for $\rho_{l0} < 0$.

However, it is easy to note that in this case (4.17), (4.18) should be altered as follows:

$$\rho_l = \{s - \rho_{l0} - \operatorname{th}[s - \operatorname{arth} \rho_{l0}]\} \times \chi\,]-s \operatorname{sign} \rho_{l0}] + \rho_{l0}$$
$$\rho_k = \left\{\frac{\rho_{k0}}{\rho_{*0} \operatorname{ch}[s - \operatorname{arth} \rho_{l0}]} - \rho_{k0}\right\} \times \chi[-s \operatorname{sign} \rho_{l0}] + \rho_{k0} \tag{4.20}$$

The formulas obtained above solve the first part of the problem of one-dimensional loading of a body with initial elastic microstrains, namely, they express the local plastic deformations in terms of the macroscopic stresses and the local value of the elastic microstrains, i.e. determine one of the possible realizations of plastic deformation in a polycrystal. In order to find the macroscopic plastic deformation it is necessary to solve also the second part of the problem, namely, to calculate the mathematical expectation of plastic deformation knowing the distribution law of the initial microstrains (3.2) and their dependence on the local plastic deformations expressed by (4.17), (4.18), (4.20).

Let us use the known expression for the mathematical expectation of quantities dependent on the components of a random vector

$$\langle E_j^p \rangle = \int\!\!\int\!\!\int\!\!\int\!\!\int \varepsilon_j^p p(\varepsilon_k^\circ)\, d\varepsilon_1^\circ d\varepsilon_2^\circ \ldots d\varepsilon_5^\circ \tag{4.21}$$

Substituting (3.3) here instead of $p\ (\varepsilon_k^\circ)$, and

$$\varepsilon_k^\circ = \frac{\tau}{2G}\rho_k^\circ, \qquad \varepsilon_j^p = \frac{\tau}{\alpha}\rho_j \tag{4.22}$$

in place of ε_k° and ε_j^p, and taking into account that according to the assumption (2.3)

$$\rho_0^2 = \sqrt{\rho_{10}^2 + \rho_{20}^2 + \cdots + \rho_{50}^2} = 1 \tag{4.23}$$

we obtain

$$\langle E_j^p \rangle = \frac{1}{(2\pi)^{5/2}} \frac{1}{b^5} \frac{1}{\alpha} \int\!\!\int\!\!\int\!\!\int\!\!\int \tau\rho_j \exp\frac{-\tau^2}{2b^2} d(\tau\rho_{10})\, d(\tau\rho_{20}) \ldots d(\tau\rho_{50}) \tag{4.24}$$

Integration is performed here over all $\tau\rho_{k0}$ within the limits 0 to ∞; b^2 is the variance (dispersion) of the intensity of the tensor of the dry friction force (3.7).

Spherical coordinates in the five-dimensional space ρ_{k0} should be used in an actual calculation of the integrals (4.24) by putting

$$\rho_{1,0} = \cos\theta_1,\ \rho_{2,0} = \sin\theta_1\cos\theta_2,\ \rho_{3,0} = \sin\theta_1\sin\theta_2\cos\theta_3$$
$$\rho_{4,0} = \sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4,\quad \rho_{5,0} = \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 \tag{4.25}$$

Here τ plays the part of a radius-vector in this spherical coordinate system. The new coordinates have the following ranges of variation:

$$0 \leqslant \tau \leqslant \infty,\ 0 \leqslant \theta_1 \leqslant \pi,\ 0 \leqslant \theta_2 \leqslant \pi,\ 0 \leqslant \theta_3 \leqslant \pi,\ 0 \leqslant \theta_4 \leqslant 2\pi \tag{4.26}$$

within whose limits there exists a one-to-one correspondence between all points of the five-dimensional Euclidean space, and the curvilinear coordinates. The Jacobian of this coordinate system is

$$J = \tau^4 \sin^3\theta_1 \sin^2\theta_2 \sin\theta_3 \tag{4.27}$$

and therefore, (4.24) in the coordinates τ, θ_k is written thus:

$$\langle E_j^p \rangle = \frac{1}{(2\pi)^{5/2}} \frac{1}{b^5} \frac{1}{\alpha} \times \int_0^\infty \int_0^\pi \int_0^\pi \int_0^\pi \int_0^{2\pi} \rho_j \exp\frac{-\tau^2}{2b^2} \tau^5 \sin^3\theta_1 \sin^2\theta_2 \sin\theta_3\, d\tau\, d\theta_1 \ldots d\theta_4 \tag{4.28}$$

This expression can be used to calculate the mathematical expectations of plastic deformations in both the particular one-dimensional loading case considered above, and also for other kinds of loadings. In conclusion, let us determine the mathematical expectations of E_{ij}^p in the general case when the inequality $\rho_0 \leqslant 1$ is taken instead of the equality $\rho_0 = 1$.

Then E_0 turns out to be a random parameter independent of T, whose upper bound is the inequality $E_0 \leqslant T/2G$. This latter bound is mathematically equivalent to the demand that the joint distribution density of E_k and T be zero in the domain $E_0 > T/2G$. Again introducing spherical coordinates in the five-dimensional space E_k°, we obtain

$$\langle E_k^p \rangle = \frac{2}{9\pi} \frac{1}{a^5 b^5} \int_0^\infty \tau^4 \exp\frac{-\tau^2}{2b^2} d\tau \int_0^{\tau/2G} \varepsilon_0^4 \exp\frac{-\varepsilon_0^2}{2a^2} d\varepsilon_0 \times \tag{4.29}$$

$$\times \int_0^\pi \sin^3\theta_1\, d\theta_1 \int_0^\pi \sin^2\theta_2\, d\theta_2 \int_0^\pi \sin\theta_3\, d\theta_3 \int_0^{2\pi} \varepsilon_k^p\, d\theta_4$$

To obtain (4.28) from (4.29) it is necessary to put formally

$$p(\varepsilon_0) = \frac{2}{3}\frac{1}{\sqrt{2\pi}}\frac{\varepsilon_0^4}{a^5}\exp\frac{-\varepsilon_0^2}{2a^2} \div \delta\left(\frac{\tau}{2G} - \varepsilon_0\right) \qquad (4.30)$$

Here δ is the Dirac function

$$\delta(x) = \begin{cases}\infty & (x = 0), \\ 0 & (x \neq 0),\end{cases} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \qquad (4.31)$$

since precisely such a substitution means that only the value ε_0 coincident with $\tau/2G$ is probable.

5. Mathematical expectation of the plastic deformation of a polycrystal devoid of initial microstresses. Let us start an investigation of the results of the theory with the simplest particular case, namely, let us assume that there are no initial microstresses. In this case the ε_l^p will be functions only of the scalar parameter τ in conformity with which

$$\langle E_l^p \rangle = \int_0^\infty \varepsilon_l^p(\tau)\, p(\tau)\, d\tau \qquad (5.1)$$

where $p(\tau)$ is defined by (3.9).

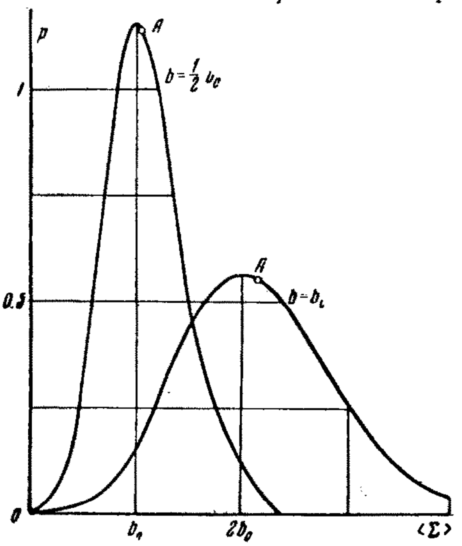

Fig. 1

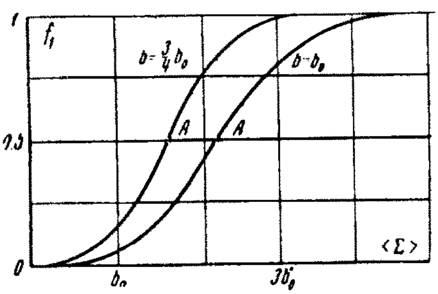

Fig. 2

The solution obtained in the preceding Section remains valid even for $\rho_{k0} = 0$, hence $s_{10} = +1$, $s_{20} = -1$. Therefore

$$\rho_l = (s - \operatorname{sign} s_0)\chi\,[(s - s_0)\operatorname{sign} s_0], \quad \rho_k = 0 \quad (k \neq l) \qquad (5.2)$$

Returning to the initial notations here, and putting $s_0 = +1$ for definiteness, i.e. considering $s > 0$, we obtain

$$\varepsilon^p = \frac{1}{\alpha}(\langle\Sigma\rangle - \tau)\,\chi(\langle\Sigma\rangle - \tau) \qquad (5.3)$$

For simplicity in writing the subsequent formulas, the subscripts l on the ε_l^p and $\langle\Sigma_l\rangle$ have been omitted here.

Substituting (5.3) into (5.1), we arrive at the following formula which expressed the dependence between the mathematical expectetions of the stresses and plastic deformation in the one-dimensional loading case

$$\langle E^p\rangle = \frac{1}{\alpha}\int_0^{\langle\Sigma\rangle}(\langle\Sigma\rangle - \tau)\,p(\tau)\,\alpha\tau \tag{5.4}$$

Differentiating (5.4) with respect to $\langle\Sigma\rangle$, we have

$$\frac{d\langle E^p\rangle}{d\langle\Sigma\rangle} = \frac{1}{\alpha}\int_0^{\langle\Sigma\rangle} p(\tau)\,d\tau \tag{5.5}$$

It hence follows that

$$\langle E^p\rangle = \frac{1}{\alpha}\int_0^{\langle\Sigma\rangle} d\tau \int_0^{\tau} p(\xi)\,d\xi \tag{5.6}$$

i.e the mathematical expectation of the plastic deformation in the absence of initial microstresses turns out to be proportional to the double integral of the density distribution of the yield point. Shown in Figs. 1–3 are curves picturing

$$p(\langle\Sigma\rangle) = \frac{d^2\langle E^p\rangle}{d\langle\Sigma\rangle^2}\alpha, \qquad f_1(\langle\Sigma\rangle) = \alpha\frac{d\langle E^p\rangle}{d\langle\Sigma\rangle}$$

$$\langle\Sigma\rangle = f_2(\alpha\langle E^p\rangle)$$

for two different values of the variance b^2. Values corresponding to $\langle\Sigma\rangle = \langle T\rangle$, i.e. the mathematical expectation of the yield point which, as has already been pointed out earlier, for micro-inhomogeneous bodies, should be considered as the effective yield point of the material, are marked by points A on these curves. There results from (5.5), (5.4) that the curve $\langle\Sigma\rangle = f_2(\alpha\langle E^p\rangle)$ has the asymptote

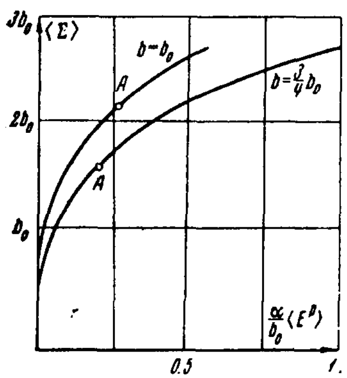

Fig. 3

$$\langle\Sigma\rangle = \alpha\langle E^p\rangle + \langle T\rangle$$

The slope of this line is defined by the parameter α which plays the part of the hardening modulus, and its initial ordinate equals the mathematical expectation of the yield point. The curves in Fig. 3 are sufficiently similar to the experimental curves $\langle\Sigma\rangle = f(\langle E^p\rangle)$ for materials which do not have flow plateaus, which indicates that the distribution law (3.8) will be a satisfactory approximation to the true distribution law of the local yield points in actual polycrystalline materials of the mentioned kind.

6. Mathematical expectation of plastic deformation of a polycrystal with initial microstresses (simplified modification of the theory). Underlining the preceding discussions was the assumption that the components of the initial microstrains E_k° are independent random variables subject to a normal distribution law. After a number of discussions and computations, this resulted in (4.28) and (4.29), governing the averaged plastic deformations under conditions (2.3) or (2.2). Let us consider a simplified modification of this theory, based on replacing the distribution density of the dry friction intensity $p(\tau)$ (3.8), and the density of the intensity of the elastic initial microstrains $p(\varepsilon_0)$ (4.30) approximately by the expressions

$$p(\tau)=\frac{2}{3}\frac{1}{\sqrt{2\pi}}\frac{\tau^4}{b^5}\exp\frac{-\tau^2}{2b^2}\approx\delta(\tau-\langle T\rangle) \tag{6.1}$$

$$p(\varepsilon_0)=\frac{2}{3}\frac{1}{\sqrt{2\pi}}\frac{\varepsilon_0^4}{a^5}\exp\frac{-\varepsilon_0^2}{2a^2}\approx\delta(\varepsilon_0-\langle E_0\rangle)$$

Utilization of the approximate formulas (6.1) means that in calculating the random variables dependent on τ and ε_0 all the realizations of these latter quantities will be identified in a first approximation, with their mathematical expectations. Substituting (6.1) in place of the corresponding members in the integrands in (4.29) or (4.28), we obtain

$$\langle E_k^p\rangle=\iiiint\!\!\iint\varepsilon_k^p\delta(\tau-\langle T\rangle)\,\delta(\varepsilon_0-\langle E_0\rangle)\sin^3\theta_1\sin^2\theta_2\sin\theta_3\,d\tau\,d\varepsilon_0\,d\theta_1\ldots d\theta_4 \tag{6.2}$$

This formula is substantially simpler than the general relationships (4.29). Nevertheless it permits development of a number of interesting features of the proposed theory.

Let us use the results obtained earlier for the one-dimensional loading case, and substituting them into (6.2) we obtain

$$\langle E_k^p\rangle=0 \quad \text{for } k\neq l,$$

$$\langle E_l^p\rangle=\frac{3}{4}\int_0^\pi\varepsilon_e^p\sin^3\theta_1\,d\theta_1$$

where ε_l^p is defined by (4.4) and (4.7).

Presented in Fig. 4 are typical curves for the development of local plastic strains (the graphs are constructed as a function of the angle θ_1 for $2G\varepsilon_0/\tau=1$). Attention should be turned to the

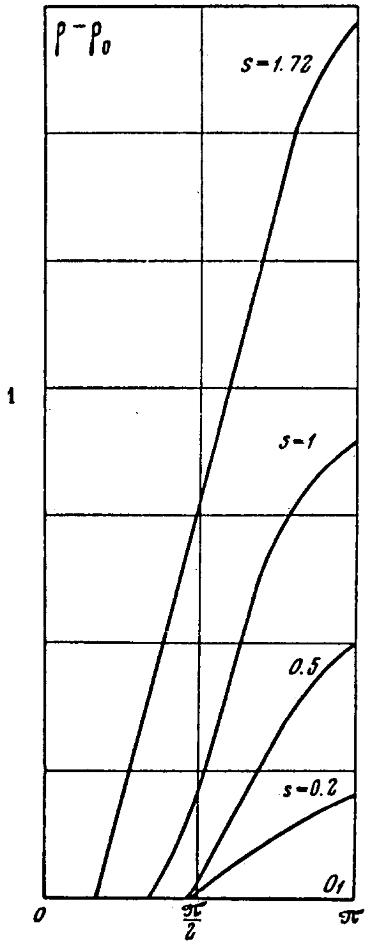

Fig. 4

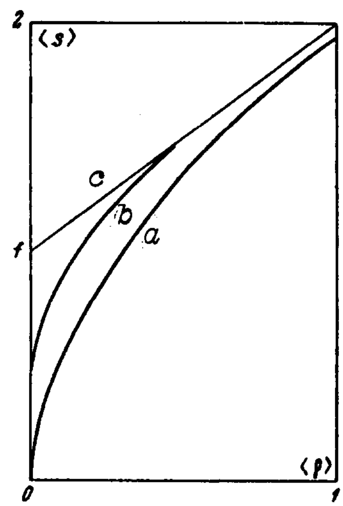

Fig. 5

fact that the dependence of ε_l^p on θ_1 turns out to be almost linear, which considerably simplifies the evaluation of $\langle E_l^p\rangle$.

Shown in Fig. 5 are graphs of $\langle \Sigma \rangle = f(\langle E^p \rangle)$ for three cases

$$(a)\ \frac{2G\langle E_0\rangle}{\langle T\rangle}=1, \qquad (b)\ \frac{2G\langle E_0\rangle}{\langle T\rangle}=\frac{1}{2}, \qquad (c)\ \frac{2G\langle E_0\rangle}{\langle T\rangle}=0$$

It is seen from these curves that the quantity $\langle E_0\rangle$ (the mathematical expectation of the intensity of initial microstrains) substantially influences the nature of the hardening. Growth of $\langle E_0\rangle$ (for fixed values of $\langle T\rangle$) results in an earlier appearance of plastic deformations, and their slower tendency to the asymptote. Correspondingly, the greater the $\langle E_0\rangle$, the longer does the process of stabilization of the hysteresis loop under cyclic deformation turn out to be (Fig. 6).

Pictured in Fig. 7 are the flow surfaces corresponding to cases (*b*) and (*c*).

In the absence of initial microstrains and the simplifying assumptions utilized above, the flow surface is shifted as a solid whole according to the fundamental modification of translation flow theory [8]. In the presence of initial elastic microstrains, the shape and size of the flow boundaries turn out to be dependent on the previous plastic

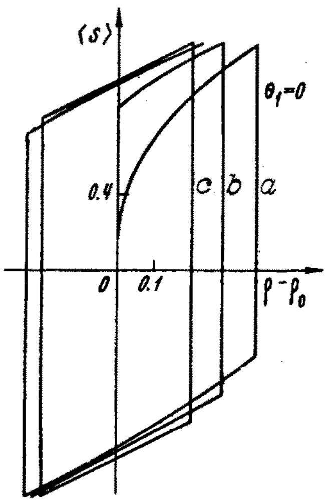

Fig. 6

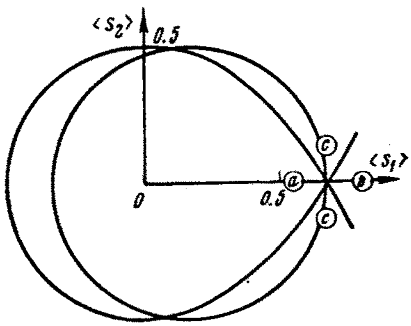

Fig. 7

deformations, whereupon corner points form on the flow boundaries (yield surface) during plastic deformation. However, these points are not as acute as in slip theory [9]. Moreover, an analysis of the computations performed for the considered modification of the theory showed that in the neighborhood of the corner points, three zones *a*, *b* and *c* must be distinguished (Fig. 7). Zone *a* is the elastic strain domain. The direction of the plastic deformation increment in zone *b* is independent of the stress increment just as is assumed in flow theory. However, this latter does not hold in zone *c*, where the direction of the vector $d\langle E_k{}^p\rangle$ depends not only on $\langle \Sigma_k\rangle$, but also on $d\langle \Sigma_k\rangle$. The zones *c* turn out to be transitions, therefore: the regularities of plastic deformation gradually change in them from those which are assumed in flow theory to those in elasticity theory (depending on the direction of the vectors $d\langle \Sigma_1\rangle$, $d\langle \Sigma_2\rangle$). This latter makes experimental detection of corner points of the considered kind quite difficult. In case (*a*), when we have $2G\langle E_0\rangle/\langle T\rangle = 1$, the plastic deformations change with any change in stress, and the concept of the flow surface, in the exact sense of this word, vanishes, as has already been remarked before.

7. Conclusion. The curves presented above, which express the dependence of

$\langle E^p{}_l \rangle$ on $\langle \Sigma_l \rangle$, were constructed for three different relationships between $2G \langle E_0 \rangle$ and $\langle T \rangle$, starting from a simplified theory based on replacement of all realizations of the random variables E_0 and T by their mathematical expectations. The curves mentioned yield a qualitative representation of the influence of initial elastic microstrains on the connection between the macroscopic stresses and the plastic strains. A more exact judgement of this influence could be obtained without involving the mentioned simplification, but by utilizing directly the general formulas (4.28), (4.29), which would however involve more tedius calculations. The appropriate information will be presented in the authors' paper being prepared for publication, which is devoted to applying the considered theory to cyclic loadings. It is easy to forsee that for a monotone one-dimensional loading, the general theory will yield results which will differ from the approximate results presented above as follows.

1. The flow boundary (yield surface), in the exact sense of the word, will be absent for any relationship between $\langle E_0 \rangle$ and $\langle T \rangle$, and not only for $2G \langle E_0 \rangle = \langle T \rangle$, as has been obtained above.

The concept of flow boundary can be introduced in the general theory only by assigning some tolerance Δ on the quantity $\langle E^p \rangle$ starting with which the macroscopic plastic deformations are taken into account, as has been done in practice.

2. As the ratio $2G \langle E_0 \rangle : \langle T \rangle$ increases, the conditional yield point $\langle \Sigma \rangle$ (i.e. the value of the macroscopic stresses for which $\langle E^p \rangle$ reaches the value Δ) is lowered, and the curve $\langle \Sigma \rangle = f(\langle E^p \rangle)$ tends more slowly to its asymptote. Both these effects are qualitatively retained, however, in sharper form in the approximate theory expounded above.

In the authors' opinion, the proposed quasistatic theory of plasticity is of greatest interest for the analysis of the strain picture under cyclic loads, where it permits an interpretation of a number of fine effects observed in tests, and it can be utilized for the classification of materials (from the viewpoint of their reaction under cyclic loads). Moreover, this theory can turn out to be useful even for an approximate estimate of the influence of initial and strain microstresses on the strength of materials since it discloses the possibility of assessing the mathematical expectation and variance of the intensity of initial microstresses, as well as the changes in these quantities during plastic deformation (on the basis of experimentally observed curves connecting the macroscopic stresses and strains.

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Translated by M. D. F.

ON St.VENANT FLEXURE INCLUDING COUPLE STRESSES

PMM Vol.32, №5, 1968, pp. 923-929

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Introduction. We consider the St. Venant flexure problem for beams of narrow rectangular cross section, that is under the assumption of plane stress, for two reasons. The first of these is that it is possible to give an exact solution in closed form for this problem including significant effects of couple stresses. The second reason is that this problem may be considered as a special case of the problem of deriving two-dimensional shell theory from three-dimensional elasticity theory in the iterative manner which has been presented for the general case in September 1967 in Kopenhagen at the Second Symposium on Shell Theory of the International Union of Theoretical and Applied Mechanics (IUTAM).

Formulation of the problem. Appropriate differential equations are three equilibrium equations

$$\sigma_{xx,x} + \sigma_{yx,y} = 0, \quad \sigma_{xy,x} + \sigma_{yy,y} = 0, \tag{1a, b}$$

$$\tau_{x,x} + \tau_{y,y} + \sigma_{xy} - \sigma_{yx} = 0 \tag{1c}$$

three compatibility equations (*)

$$e_{xx,y} - e_{yx,x} + k_x = 0, \quad e_{xy,y} - e_{yy,x} + k_y = 0, \quad k_{x,y} - k_{y,x} = 0 \tag{2a, b, c}$$

and six stress strain relations which are here been taken in the form

$$E_x e_{xx} = \sigma_{xx} - \nu_x \sigma_{yy}, \quad E_y e_{yy} = \sigma_{yy} - \nu_y \sigma_{xx} \tag{3a, b}$$

$$2G_x e_{xy} = \sigma_{xy}, \quad 2G_y e_{yx} = \sigma_{yx} \tag{3c, d}$$

$$c^2 \Gamma_x k_x = \tau_x, \quad c^2 \Gamma_y k_y = \tau_y \tag{3e, f}$$

with $(\nu_x / E_x = \nu_y / E_y)$.

The system (1) to (3) is to be solved in the rectangular region $|y| \leqslant c$, $|x| \leqslant L$ subject to boundary conditions

$$y = \pm c, \quad \sigma_{yx} = \sigma_{yy} = \tau_y = 0 \tag{4}$$

*) which are a consequence of strain displacement relations

$$e_{xx} = u_{,x}, \; e_{yy} = v_{,y}, \; k_x = \psi_{,x}, \; k_y = \psi_{,y}, \; e_{xy} = v_{,x} - \psi, \; e_{yx} = v_{,y} + \psi$$